Lecture 28

Last time:

Prop: Suppose that $X_i = N(\mu_i, \sigma_i^2)$ a normal RV. Then $\sum_i X_i = N(\sum \mu_i, \sum \sigma_i^2)$.

Recall that Y is called lognormal with params μ, σ if $Y = e^X$ for $X = N(\mu, \sigma^2)$.

Ex: Suppose that $S(n)$ is the price of a certain security at the end of n weeks from right now. A popular model for the evolution of these prices assumes that the price ratios $S(n)/S(n-1)$, $n \geq 1$.

Assuming the model has parameters $\mu = 0.0165$ and $\sigma = 0.0730$, what is the prob that the price of the security at the end of two weeks is higher than it is today?

Solution: Observe that $S(2) > \underbrace{S(0)}_{\text{price r.n.}}$ iff $\dfrac{S(2)}{S(0)} > 1$.

So we want $P\left(\frac{S(2)}{S(0)} > 1\right) = P\left(\frac{S(2)}{S(1)} \cdot \frac{S(1)}{S(0)} > 1\right)$

$$= P\left(\log\left(\frac{S(2)}{S(1)}\right) + \log\left(\frac{S(1)}{S(0)}\right) > 0\right).$$

Now, $\log\left(\frac{S(n)}{S(n-1)}\right) = N(0.0165, (0.0730)^2)$

So $\log\left(\frac{S(2)}{S(1)}\right) + \log\left(\frac{S(1)}{S(0)}\right) = N(\underbrace{2(0.0165)}_{0.0330}, 2(0.0730)^2)$

Hence $P\left(\frac{S(2)}{S(0)} > 1\right) = P\left(N(0.0330, 2(0.0730)^2) > 0\right)$

$$= P\left(Z > \frac{-0.0330}{(0.0730)\sqrt{2}}\right) = 0.6254.$$

Some Examples of Sums of independent RV's in the Discrete case

<u>Ex</u>: Sums of independent Poisson RV's.

Recall that X is Poisson with parameter λ iff X has a probability mass function given by

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

Suppose that X_1 and X_2 are independent Poisson RV's with parameters λ_1, λ_2 respectively. Then, for any n,

$$p_{X_1+X_2}(n) = P(X_1+X_2 = n) = \sum_{k=0}^{n} P(X_1 = k, X_2 = n-k)$$

$$= \sum_{k=0}^{n} P(X_1 = k)\, P(X_2 = n-k) \leftarrow \text{by indep.}$$

$$= \sum_{k=0}^{n} \left(\frac{e^{-\lambda_1}\lambda_1^k}{k!}\right)\left(\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\right)$$

$$= e^{-(\lambda_1+\lambda_2)} \sum_{k=1}^{n} \frac{\lambda_1^{k}\,\lambda_2^{n-k}}{k!\,(n-k)!}$$

multiply by $\frac{n!}{n!} = 1$.

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=1}^{n} \frac{n!}{k!\,(n-k)!} \lambda_1^{k} \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=1}^{n} \binom{n}{k} \lambda_1^{k} \lambda_2^{n-k}$$

Binomial theorem.

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^{n}$$

Hence $X_1 + X_2$ is again a poisson RV with parameter $\lambda_1+\lambda_2$.

By induction on the # of RV's, if $X_1, \dots, X_m$ are independent Poisson RV's with parameters λ_i, $i=1,\dots m$. Then $\sum_{i=1}^{m} X_i$ is again a Poisson RV with parameter $\sum_{i=1}^{m} \lambda_i$.

<u>Ex</u>: Sums of binomial RV's with fixed p.

Suppose that $X = Bin(n, p)$ $Y = Bin(m, p)$.
By definition X and Y are sums of independent Bernoulli trials. By independence, $X + Y = Bin(n+m, p)$
(See the text for a formal proof).

Conditional Distributions

Discrete Case:

Recall that for events E and F,
$$P(E|F) = \frac{P(EF)}{P(F)}, \text{ if } P(F) > 0.$$

Suppose that X and Y are discrete RV's.

Def: The conditional pmf of X given $Y=y$ is defined

$$p_{X|Y}(x|y) = P(X=x | Y=y)$$
$$= \frac{P(X=x, Y=y)}{P(Y=y)} = \boxed{\frac{p(x,y)}{p_Y(y)}}$$

for all y s.t. $p_Y(y) > 0$. (Here, $p(x,y)$ is the joint prob. mass function, and $p_Y(y)$ is the marginal probability of Y.)

We can derive an expression for the "cumulative conditional probability of X given $Y=y$" as

$$F_{X|Y}(x|y) = P(X \le x | Y=y)$$
$$= \sum_{a \le x} p_{X|Y}(a|y)$$

(Note that these definitions are symmetric in X and Y).

Observation: If X and Y are independent, then

$$p_{X|Y}(x|y) = \frac{P(X=x,\ Y=y)}{P(Y=y)} = \frac{P(X=x)\cancel{P(Y=y)}}{\cancel{P(Y=y)}}$$

$$= P(X=x) = p_X(x).$$

Ex: Let X_1, X_2 be independent Poisson RV's with parameters λ_1, λ_2 resp. Find the conditional distribution of X_1 given that $X_1 + X_2 = n$.

Soln:

$$p_{X_1|X_1+X_2}(k|n) = P(X_1 = k \mid X_1 + X_2 = n)$$

$$= \frac{P(X_1 = k,\ X_1 + X_2 = n)}{P(X_1 + X_2 = n)}$$

$$= \frac{P(X_1 = k,\ X_2 = n-k)}{P(X_1 + X_2 = n)}$$

$$= \frac{P(X_1 = k)\,P(X_2 = n-k)}{P(X_1 + X_2 = n)}. \quad \left(\text{by indep of } X_1, X_2\right).$$

Now, from earlier, $X_1 + X_2$ is Poisson with parameter $\lambda_1 + \lambda_2$, so

$$p_{X_1|X_1+X_2}(k|n) = \left(\frac{\cancel{e^{-\lambda_1}}\lambda_1^k}{k!}\right)\left(\frac{\cancel{e^{-\lambda_2}}\lambda_2^{n-k}}{(n-k)!}\right)\left(\frac{n!}{\cancel{e^{-(\lambda_1+\lambda_2)}}(\lambda_1+\lambda_2)^n}\right)$$

$$= \frac{n!}{k!\,(n-k)!}\ \frac{\lambda_1^k\,\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n}$$

$$= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

Now, observe that

$$1 - \frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{(\lambda_1+\lambda_2)-\lambda_1}{\lambda_1+\lambda_2} = \frac{\lambda_2}{\lambda_1+\lambda_2} \quad (>1)$$

and so

$$P_{X_1|X_1+X_2}(k|n) = \underbrace{\binom{n}{k} p^k (1-p)^{k}}_{\text{binomial}}, \text{ where } p = \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

<u>Next</u>: Conditional distributions, the continuous case:

Suppose that X and Y are jointly distributed with density $f(x,y)$. Then we define the conditional probability of X given Y as

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}, \quad f_Y(y) > 0.$$